Minimizing the Solid Angle Sum of Orthogonal Polyhedra and Guarding them with $\frac{\pi}{2}$-Edge Guards

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Abstract

We give a characterization for the orthogonal polyhedron in $\mathbb{R}^3$ that minimizes the sum of its internal solid angles, and prove that its minimum angle sum is $(n-4)\pi$ and their maximum angle sum is $(3n-24)\pi$. We generalize to $\mathbb{R}^3$ the well-known result that in an orthogonal polygon with n vertices, $(n+4)/2$ of them are convex and $(n-4)/2$ of them are reflex. We define a vertex of a polyhedron to be convex on the faces if it is convex or straight in all the faces where it participates, and to be reflex on the faces otherwise. If a polyhedron with n vertices and genus g has k vertices of degree greater than 3 (in its 1-skeleton), we prove that it has $(n+8-8g+3k)/2$ vertices that are convex on the faces and $(n-8+8g-3k)/2$ vertices that are reflex on the faces. Finally, we prove that if the orthogonal polyhedron has k_4 vertices of degree 4, k_6 vertices of degree 6, genus g and h_m holes on its faces, then we can guard it using at most $(11e-k_4-3k_6-12g-24h_m+12)/72$ $\frac{\pi}{2}$-edge guards (i.e., having a visibility angle of $\pi/2$ towards the interior of the polyhedron), improving the bound given by Viglietta et al in [14] for open edge guards.

1 Introduction

In the plane, to measure the interior angle of a polygon at a vertex v, we usually consider a small enough circle centered at v and not containing any other vertices of the polygon, measure the length of the portion of the circle that lies inside the polygon, and then divide it by the radius. In this way, we can have angles that vary between 0 and 2π. It has been well-known since antiquity that the sum of the angles of a triangle is π. Since a simple polygon of n vertices can be partitioned into $n-2$ triangles using diagonals, the sum of the internal angles of a polygon is $(n-2)\pi$. We extend these ideas to polyhedra in $\mathbb{R}^3$.

We measure the interior solid angles of a polyhedron in a vertex v in an analogous way to the plane. We consider a small enough sphere centered at v, measure the area of the portion of the sphere that lies within the polyhedron, and then divide it by the square of the radius. In this way, we have solid angles that vary between 0 and 4π since the area of a unit sphere is 4π.

For summing interior angles in polyhedra we cannot use the same approach that was used for polygons. This approach would consist in tetrahedralizing a polyhedron and summing the solid angles of all the resulting tetrahedra. However, there exist examples of polyhedra that cannot be tetrahedralized; for example, the Schönhardt polyhedron [12]. It is also known that the sum of the solid angles of a tetrahedron can take any value between 0 and 2π [7].

These examples show that in general polyhedra, the sum of their solid angles is not constant and their vertices can have interior angles that are arbitrarily small. However, it is an interesting question to find the minimum and the maximum sums of the internal solid angles of an orthogonal polyhedron. This sum cannot be arbitrarily small because the internal solid angle of each vertex is at least $\pi/2$. We show in this paper that the lower and upper bounds for the sum of angles of an orthogonal polyhedron with n vertices are $(n-4)\pi$ and $(3n-24)\pi$ respectively. We also give the classification of the families of orthogonal polyhedra achieving these bounds.

We consider that a vertex of a polyhedron is convex on the faces if it is a convex or a straight vertex in all the faces where it participates, and it is reflex on the faces otherwise. If a polyhedron with n vertices has k vertices of degree greater than 3 in its *1-skeleton* (i.e., the set of edges and vertices of the polyhedron), we prove that it has $(n+8+3k)/2$ vertices that are convex on the faces and $(n-8-3k)/2$ vertices that are reflex on the faces.

We apply this result to address a variant of the Art Gallery Problem in orthogonal polyhedra. Most of the research on art gallery problems has been focused on polygons on the plane. For example, it is well

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known that every simple polygon with n vertices can be guarded with at most $\lfloor n/3 \rfloor$ vertex guards [4], and for orthogonal polygons $\lfloor n/4 \rfloor$ vertex guards are always sufficient to guard the polygon [8]. Estivill-Castro and Urrutia [6] showed that every orthogonal polygon can be guarded with at most $3(n-1)/8$ *orthogonal floodlights*; that is, vertex guards that have an angle of vision of $\pi/2$. Later in [1] it was proved that $(3n+4(h-1))/8$ orthogonal floodlights are always sufficient to guard an orthogonal polygon with n vertices and h holes.

For orthogonal polyhedra with e edges in $\mathbb{R}^3$, it was conjectured that $e/12$ edge guards are always sufficient to guard any polyhedron [13]. Benbernou *et al.* [14] showed that every polyhedron can always be guarded by $(11/72)e - g/6 - 1$ *open edge guards* (i.e., excluding their endpoints).

For general polyhedra, Cano *et al.* [3] showed that any polyhedron can always be guarded by $(27/32)e$ edge guards, and if the faces are all triangles the bound improves to $(29/36)e$. For general polyhedra it is conjectured that every simply connected polyhedron can be guarded with $e/6$ edge guards [13].

We say that a $\frac{\pi}{2}$-*edge guard* is a guard located on an edge of the polyhedron, occupying all the edge with an angle of vision of $\frac{\pi}{2}$. An interior point p of the polyhedron is guarded by an edge e if the segment s, described by the shortest distance between p and e, is perpendicular to e, s is contained in the visibility angle of e, and s is completely contained in the interior of the polyhedron.

The variant of the art gallery problem we address is the following: Given an orthogonal polyhedron P in $\mathbb{R}^3$, choose a minimum set of $\frac{\pi}{2}$-edge guards located on the edges of P such that any interior point of P is guarded. We prove that if P has k_4 vertices of degree 4, k_6 vertices of degree 6, genus g and h_m holes on its faces, then we can guard it using at most $(11e-k_4-3k_6-12g-24h_m+12)/72$ $\frac{\pi}{2}$-edge guards.

2 Orthogonal Polyhedra

A *polyhedron* in $\mathbb{R}^3$ is a compact set bounded by a piecewise linear manifold. A *face* of a polyhedron is a maximal planar subset of its boundary whose interior is connected and non-empty. A polyhedron is *orthogonal* if all of its faces are parallel to the xy, xz or yz planes. Faces of a polyhedron can be polygons with holes, and if the polyhedron is orthogonal, then its faces and its holes are also orthogonal. A *vertex* of a polyhedron is a vertex of any of its faces. An *edge* is a minimal positive-length straight line segment shared by two faces and joining two vertices of the polyhedron.

2.1 Vertex Characterization in Orthogonal Polyhedra

Let P be an orthogonal polyhedron in $\mathbb{R}^3$. We classify the vertices of P by its interior solid angles. A vertex x of P is classified as *1-octant* if its interior solid angle is $\pi/2$ (see Figure 1a), and *3-octant* if its interior solid angle is $3\pi/2$ (see Figure 1b). The *4-octant*, *5-octant* and *7-octant* vertices are defined in a similar way, as illustrated in Figures 1c, 1d, 1e and 1f respectively.

In an orthogonal polygon we have three kinds of vertices; convex, reflex and straight. A vertex is convex if it has an interior angle of $\pi/2$, reflex if it has an interior angle of $3\pi/2$ and straight if it has an angle of π.

We say that a vertex is *convex on the faces* if it participates on each of its incident faces as a convex or a straight vertex. Thus the 1-octant, 4-octant, and 7-octant vertices are convex on the faces. We say that a vertex is *reflex on the faces* if it participates as a reflex vertex on exactly one of its incident faces. Thus the 3-octant and 5-octant vertices are reflex on the faces. We will refer to a convex vertex on the faces (resp. reflex vertex on the faces) as a convex vertex (resp. reflex vertex) unless stated otherwise. Since we can have straight vertices on the faces of a polyhedron, we extend our concept of orthogonal polygon in order to allow them to have straight vertices, too.

The *genus* g of a connected orientable surface is the integer representing the maximum number of cuttings along non-intersecting closed simple curves without rendering the resultant manifold disconnected [9].

In our main result we use the Euler-Poincaire's formula, which states that for any polyhedron of genus g with f faces, e edges, v vertices and a total of h holes on its faces, the identity $v-e-h+f=2-2g$ holds. A proof of this theorem can be found in [11].

Next, we prove the following theorem:

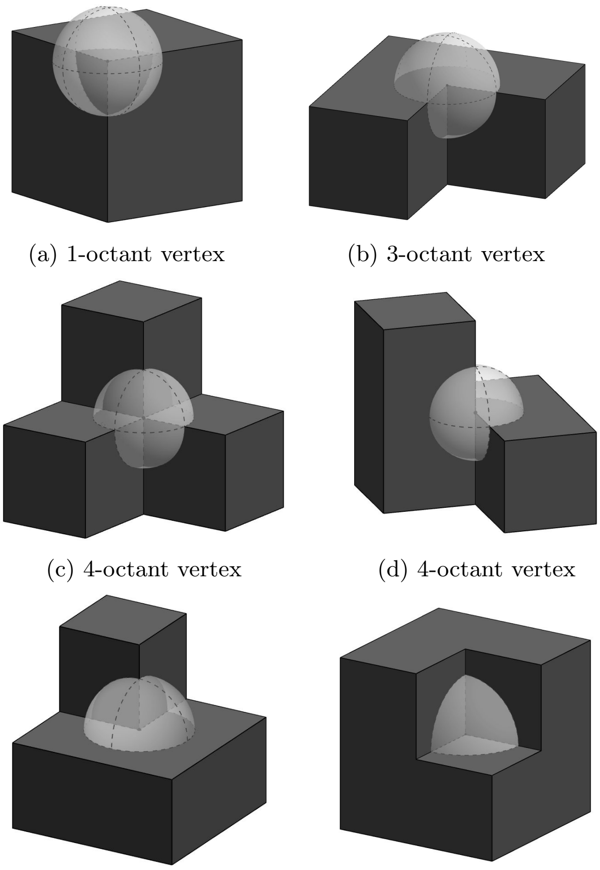
Theorem 1 *Let P be an orthogonal polyhedron in $\mathbb{R}^3$ homeomorphic to the sphere with $n=2k$ vertices and a connected and* 3*-regular 1-skeleton. Then P has $(n+8)/2$ convex vertices and $(n-8)/2$ reflex vertices.*

Proof. Since each vertex has degree 3, the number of edges e is $3k$. By Euler's formula, the number of faces f is $k+2$. The number of reflex vertices in an orthogonal polygon is $(n-4)/2$, so the number of reflex vertices on each face of P is $(V_i-4)/2$, where V_i is the number of vertices on the i^{th} face of P. Then the number of reflex vertices of P is

$$r = \sum_{i=1}^{k+2} \frac{V_i - 4}{2}. \quad (1)$$

Solving equation (1), we have

$$2r = \sum_{i=1}^{k+2} V_i - \sum_{i=1}^{k+2} 4.$$



(a) 1-octant vertex

(b) 3-octant vertex

(c) 4-octant vertex

(d) 4-octant vertex

(e) 5-octant vertex

(f) 7-octant vertex

Figure 1: Vertex classification for orthogonal polyhedra

As each vertex belongs to three faces, it is counted three times when adding up the first sum;

$$\begin{aligned} 2r &= 6k - 4(k+2) \\ r &= k - 4. \end{aligned}$$

Since $n = 2k$, $r = (n-8)/2$, and since $n = c + r$, $c = (n+8)/2$. □

This result tells us exactly the number of convex and reflex vertices of the family of orthogonal polyhedra that are connected and 3-regular in its 1-skeleton. Next, we eliminate the restriction of a connected 1-skeleton by considering the number of holes on the faces of P and including the genus of the polyhedron. We will also include the 4-octant vertices. Note that these vertices do not have degree 3, but degree 4 or 6. Some of the 4-octant vertices look like straight angles on some faces of the polyhedron.

We introduce two lemmas that will help us to incorporate the 4-octant vertices in the count of convex and reflex vertices of an orthogonal polyhedron.

Lemma 2 *In an orthogonal polygon with n vertices of which s are straight, the number of reflex vertices is $r = (n-s-4)/2$ and the number of convex vertices is $c = (n-s+4)/2$.*

Proof. Since the sum of the internal angles of a simple polygon is $\pi(n-2)$; and the angle of each convex vertex is $\pi/2$, of each reflex vertex $3\pi/2$, and of each straight angle π,

$$\pi(n-2) = \left(\frac{\pi}{2}\right)c + \left(\frac{3\pi}{2}\right)r + (\pi)s.$$

Solving for c and replacing in $n = c + r + s$ yields $n = 2r + s + 4$. Therefore, $r = (n-s-4)/2$ and $c = (n-s+4)/2$. □

If the polygon has holes, we have the next lemma.

Lemma 3 *In an orthogonal polygon P with n vertices, h holes, and a total of s straight vertices, the number of reflex vertices is $(n-s+4h-4)/2$ and the number of convex vertices is $(n-s-4h+4)/2$.*

Proof. Note that a hole is an orthogonal polygon such that its convex vertices are reflex in P, its reflex vertices are convex in P, and its straight vertices are straight in P. Thus, using Lemma 2, we have that if m is the number of vertices in P without the holes, s_m of which are straight, and each hole has n_i vertices, s_i of which are straight, then the number of reflex vertices of P is

$$r = \left(\sum_{i=1}^{h} \frac{n_i - s_i + 4}{2}\right) + \frac{m - s_m - 4}{2} = \frac{n - s + 4h - 4}{2}.$$

Then it follows automatically that the number of convex vertices in P is $(n-s-4h+4)/2$. □

Let k_3 be the vertices of degree 3, k_4 the vertices of degree 4, and k_6 the vertices of degree 6 in the 1-skeleton of a polyhedron.

We are ready to give one of our main results.

Theorem 4 *Let P be an orthogonal polyhedron in $\mathbb{R}^3$ with $n = k_3 + k_4 + k_6$ vertices and arbitrary genus g. Then P has $(n - 3(k_4+k_6) + 8g - 8)/2$ reflex vertices and $(n + 3(k_4+k_6) - 8g + 8)/2$ convex vertices.*

Proof. The number of edges e is $3k_3/2 + 2k_4 + 3k_6$. Using the Euler-Poincaré formula, the number of faces f is $k_3/2 + k_4 + 2k_6 + 2 + h - 2g$. By Lemma 3, the number of reflex vertices in P is

$$r = \sum_{i=1}^{f} \frac{V_i - s_i + 4h_i - 4}{2}, \tag{2}$$

where V_i is the number of vertices and s_i is the number of straight vertices and h_i is the number of holes on the ith face of P.

Solving Equation (2), we have

$$2r = \sum_{i=1}^{f} V_i - \sum_{i=1}^{f} s_i + \sum_{i=1}^{f} 4h_i + \sum_{i=1}^{f} 4.$$

In the first sum we count the total number of vertices: the k_3 vertices are counted three times, the k_4 vertices are counted four times and the k_6 vertices are counted six times. The second sum counts the total number of straight vertices but there are only k_4 vertices and they are counted two times. The third sum gives the total number of holes in P. Then we have

$$2r = k_3 - 2k_4 - 2k_6 - 8 + 8g. \tag{3}$$

Since $n = k_3 + k_4 + k_6$, we obtain

$$r = (n - 3(k_4 + k_6) + 8g - 8)/2.$$

Since $n = c + r$, $c = (n + 3(k_4 + k_6) + 8 - 8g)/2$. □

This generalizes the well known result that the number of convex and reflex vertices of an orthogonal polygon with n vertices are respectively $(n+4)/2$ and $(n-4)/2$, see $\mathbb{R}^2$ [10].

2.2 Minimizing the Solid Angle Sum of Orthogonal Polyhedra

Let V_i be the number of i-octant vertices, $i = 1, 3, 4, 5, 7$. The angle sum of an orthogonal polyhedron is

$$S = \frac{\pi}{2}V_1 + \frac{3\pi}{2}V_3 + 2\pi V_4 + \frac{5\pi}{2}V_5 + \frac{7\pi}{2}V_7. \tag{4}$$

Since an orthogonal polyhedron has n vertices,

$$V_1 + V_3 + V_4 + V_5 + V_7 = n. \tag{5}$$

We use the polyhedral version of Gauss-Bonnet's theorem to calculate the curvature of the polyhedron [5]. Observe that the angle deficit for 1-octant and 7-octant vertices is $\pi/2$, the angle deficit for 3-octant and 5-octant vertices is $-\pi/2$ and the angle deficit for 4-octant vertices is $-\pi$. Applying Gauss-Bonnet's theorem, where g is the genus of the polyhedron, we get

$$\frac{\pi}{2}(V_1 + V_7) - \frac{\pi}{2}(V_3 + 2V_4 + V_5) = 4\pi - 4\pi g \tag{6}$$

Multiplying (5) by π and subtracting (6) we obtain:

$$\frac{\pi}{2}V_1 + \frac{3\pi}{2}V_3 + 2\pi V_4 + \frac{3\pi}{2}V_5 + \frac{\pi}{2}V_7 = n\pi - 4\pi + 4\pi g \tag{7}$$

Adding $\pi V_5 + 3\pi V_7$ to both sides of (7) yields:

$$\frac{\pi}{2}V_1 + \frac{3\pi}{2}V_3 + 2\pi V_4 + \frac{5\pi}{2}V_5 + \frac{7\pi}{2}V_7 = \pi n - 4\pi + 4\pi g + \pi V_5 + 3\pi V_7 \tag{8}$$

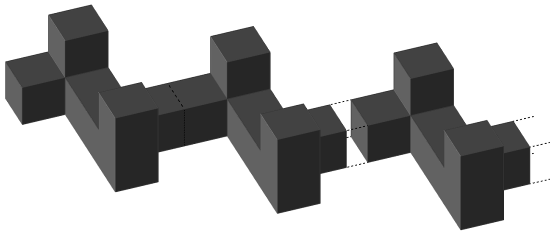


Figure 2: Family of Polyhedra that minimise the solid angle sum.

The left side of (8) corresponds to the angle sum:

$$S = \pi(n - 4 + 4g + V_5 + 3V_7) \tag{9}$$

Thus (9) is minimized when V_5 and V_7 are both equal to zero. The next result follows.

Theorem 5 *The minimum solid angle sum of orthogonal polyhedra is $(n-4)\pi$ and is achieved by polyhedra having only 1-octant, 3-octant and 4-octant vertices.*

Figure 2 shows an example that achieves the bound of Theorem 5.

The maximum solid angle sum is reached when we maximize the number of V_7 and V_5 vertices in (9). In order to do this, we observe that any orthogonal polyhedra P always has at least eight 1-vertices, and if it is not a box, it has at least eight 1-vertices and four 3-vertices, or it has ten 1-vertices and two 3-vertices. The best case arises when P has exactly eight 1-vertices and four 3-vertices. This can be achieved by *carving out of a box* a polyhedra with $m = n - 8$ vertices that minimizes the sum of its angles, as shown in Figure 3.

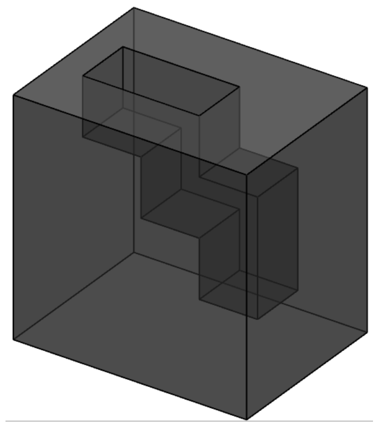


Figure 3: An Orthogonal Polyhedron that maximize its solid angle sum.

Theorem 6 *The maximum solid angle sum of orthogonal polyhedra is $(3n-24)\pi$.*

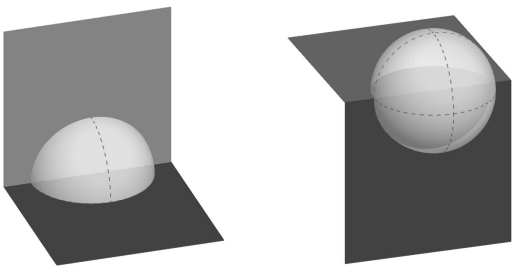
3 Guarding Polyhedra

We say that an *α-edge guard* is a guard located on an edge of a polyhedron, occupying the entire edge with an angle of vision towards the interior of the polyhedron of size α. In this section we will deal with $\alpha = \frac{\pi}{2}$.

An interior point p of a polyhedron is guarded by an α-edge guard e if the segment pr, where r is the closest point to p in e, is perpendicular to e, pr is contained in the α-visibility angle of e, and the interior of pr is contained in the interior of the polyhedron.

We apply the results obtained in the previous section to address the following variation of the Art Gallery Problem: Given an orthogonal polyhedron P in $\mathbb{R}^3$, select a set of $\frac{\pi}{2}$-edge guards located on the edges of P that guards P.

Note that P has two kinds of edges: convex edges that cover an internal solid angle of two octants, see Figure 4a, and reflex edges that cover an internal solid angle of six octants, see Figure 4b.



(a) 2-octant edge (b) 6-octant edge

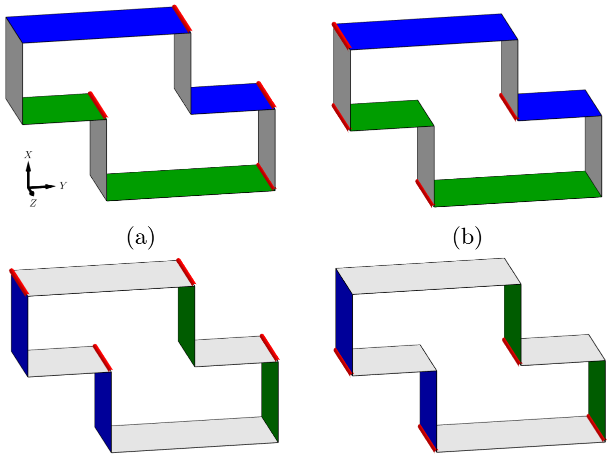
Figure 4: Types of edges in orthogonal polyhedra

It is easy to see that to guard P it is sufficient to place one $\frac{\pi}{2}$-edge guard on each convex edge, and two $\frac{\pi}{2}$-edge guards, in opposite directions, on every reflex edge. In fact, we can also guard P by applying the previous rule only to edges parallel to the $\mathcal{X}$-axis, the $\mathcal{Y}$-axis, or the $\mathcal{Z}$-axis. This follows from the results proved in [2]. For the sake of completeness we describe briefly how to prove this.

Consider all the faces of P parallel to $\mathcal{XZ}$ and $\mathcal{YZ}$ planes. We call a face of P incident to e, a *top face* f, if for any interior point q of f there is an $\epsilon > 0$ such that any point at distance less than or equal to ϵ from q, and below f belongs to the interior of P. *Right*, *bottom*, and *left* faces are defined in a similar way, see Figure 5.

Let e be an edge parallel to the $\mathcal{Z}$ axis. Given a top (bottom) face f, we call an edge of f a *right edge* if there is an $\epsilon > 0$ such that any point at distance less than or equal to ϵ from the mid-point of e, and the left of e belongs to the interior of f. A *left edge* is defined in a similar way. Given a right (left) face f, the *top* and *bottom edges* are defined similarly to the left and right edges, see Figure 5.

We define the placement rules for $\frac{\pi}{2}$-edge guards at the edges of P parallel to the $\mathcal{Z}$ axis, as follows: In the *top-right* rule at each right edge of each top face of P, and at each top edge of each right face of P we place a $\frac{\pi}{2}$-edge guard whose angle of illumination covers the interval of directions $\frac{3\pi}{2}$ to 2π. We define three extra rules, the *top-left* rule, *bottom-right* rule, and *bottom-left* rule in a similar way by rotating our polyhedra 90, 180 and 270 degrees with respect to the $\mathcal{Z}$-axis, and applying the top-left rule to the polyhedron obtained from P after applying these rotations.



(a) (b) (c) (d)

Figure 5: Figures (a) and (b) show top faces in blue and bottom faces in green. Figures (c) and (d) show left faces in blue and right faces in green. Figures (a), (b), (c) and (d) show right, left, top and bottom edges respectively.

Now we prove the following Lemma:

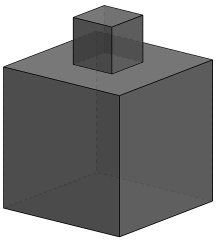
Lemma 7 *Let P be an orthogonal polyhedron with genus g and h holes on its faces. Then P can be guarded by the $\frac{\pi}{2}$-edge guards placed by any of the following rules: top-right, top-left, bottom-right and bottom-left.*

Proof. We prove our result for the top-right rule, the other rules can be proved in a similar fashion. Let p be a point in P and let β be the plane parallel to the $\mathcal{XY}$ plane containing p. Let Q be the intersection of P with β. Q consists of a set of orthogonal polygons contained in β. It is straightforward to see that the top right rule places $\frac{\pi}{2}$-vertex guards as in the top-right illumination rule in [2] which illuminates, and thus guards p. Our result follows. $\square$

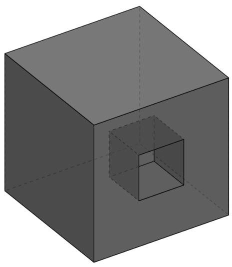
Some faces of an orthogonal polyhedron P may have holes in them. When these holes appear, the 1-skeleton of P may become disconnected, for an example see Figure 3. In that example we "carved out" an orthogonal polyhedron H from a box in the middle of one of its faces, call it f. Observe that the k-vertices of H became $8-k$-vertices in P, except for those lying in f, in that case 1-octant vertices of H became 3-octant vertices of P, and 3-octant vertices of H become 1-octant vertices of P, (*i.e.* the convex vertices become reflex and the reflex vertices become convex), see also Figure 6b. Observe that at least four of the vertices of H in f are reflex, and that two of the edges incident to them, are convex, and one is reflex. Thus our guarding rules place only four edge guards on these edges. This will be used in the proof of our next Theorem, as this will allow us to save four edges per each hole in which we carved an orthogonal polyhedron (in that proof we place five edges in the edges of a reflex vertex of degree three).

There is a second case in which the 1-skeleton of P becomes disconnected, and this happens when instead of carving out an orthogonal polyhedron H, we kind of "glue" it in the middle of a face f of P, see Figure 6a. In this case it is easy to see that when we apply the guarding rules to P described above, the points of P in H will be guarded by edges in H, and the edges in $P-H$ can be guarded with edges in the 1-skeleton of $P-H$. This implies that the edges of H in f can be considered as convex edges when applying the guarding rules described above. Thus we save at least four edge guards, one for each reflex edge of H in f.

In both cases we save at least four guards per hole.



(a)



(b)

Figure 6: (a) Two "glued" orthogonal polyhedron. (b) An orthogonal polyhedron carved out of another one.

Theorem 8 *Let P be an orthogonal polyhedron with n vertices, k_4 of them are of degree 4, k_6 of degree 6, e edges, genus g and h_m holes in the faces of P. Then $(11e-k_4-3k_6-12g-24h_m+12)/72$ $\frac{\pi}{2}$-edge guards are always sufficient to guard the interior of P.*

Proof. First we look at the type of vertices of the polyhedron P, and describe the number of convex and reflex edges that each kind of vertex is incident to.

Each 1-octant vertex is incident to three convex edges. Each 3-octant vertex is incident to two convex edges and one reflex edge. Each 4-octant vertex with degree four, is incident to two convex edges and two reflex edges. Each 4-octant vertex with degree six, is incident to three convex edges and three reflex edges. Each 5-octant vertex is incident to one convex edge and two reflex edges. Finally, each 7-octant vertex is incident to three reflex edges.

By the Theorem 4, P has $c = (n+3(k_4+k_6)-8g+8)/2$ convex vertices and $r = (n-3(k_4+k_6)+8g-8)/2$ reflex vertices. Note that according to our definition, 4-octant vertices, whether they have degree four or six are convex. Then, P has $c = (n+k_4+k_6-8g+8)/2$ convex vertices, k_4 4-octant vertices of degree four, k_6 4-octant vertices of degree six, and $r = (n-3(k_4+k_6)+8g-8)/2$ reflex vertices.

In the worst case every convex vertex is adjacent to three reflex edges, every 4-octant vertex of degree four is adjacent to two reflex edges and two convex edges, every 4-octant vertex of degree six is adjacent to three reflex edges and three convex edges, and every reflex vertex is incident to two reflex edges and one convex edge.

If we place guards on every edge of P then, we have $(6c+6k_4+9k_6+5r)/2$ $\frac{\pi}{2}$-edge guards in total. We can divide the number of $\frac{\pi}{2}$-edge guards by three and four, since it is sufficient to choose one of the three axis directions, and we only need to choose the smallest of the four guarding rules used in this direction, then we obtain $(6c+6k_6+9k_6+5r)/24$. Substituting c and r in the above equation, we have a total of $(11n+3k_4+9k_6+8)/48$ $\frac{\pi}{2}$-edge guards.

As P has h_m holes on its faces, and for each of them we save four edge guards we conclude that the total number of $\frac{\pi}{2}$-edge guards in P is $(11n+3k_4+9k_6-8g-16h_m+8)/48$. If we substitute $n = (2e-k_4-3k_6)/3$ in the number of $\frac{\pi}{2}$-edge guards, then we finally obtain that $(11e-k_4-3k_6-12g-24h_m+12)/72$ $\frac{\pi}{2}$-edge guards are always sufficient to guard the interior of P.

□

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